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We show that a non-trivial topological effect breaks the conformal invariance of pure Yang-Mills theory. Thus it is possible that classic particle-like solutions exists in pure Yang-Mills theory. We find a static, non-singular solution in source-free SU(2) Yang-Mills theory in four-dimensional Minkowski space. This solution is a stable soliton characterized by non-trivial topology and imaginary A_0^a, i.e., $A_0^a A_0^a < 0$. It yields hermitian Hamilton, and finite, positive energy.

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The purpose of this paper is to study an important aspect of Yang-Mills theory: Does there exist classic, static, non-sigular solution when non-trivial topology is present? Taditionally, the pure Yang-Mills theory does not admit classical particle-like solutions [1–4]. More precisely, this famous result asserts that there exist no finite-energy non-singular solutions to the four-dimensional static Yang-Mills equations [4]. Non-existence of static solutions can be related to conformal invariance of the Yang-Mills theory, which implies that the stress-energy tensor is traceless: $T^\mu_\mu = 0 = -T_{00} + T_{ii}$, where $\mu = 0, ..., 3,\ i = 1, 2, 3$, and Minkowski metric is taken $\eta_{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}$. The positivity of energy density T_{00} requires that the sum of the principal pressures T_{ii} is everywhere positive, i.e., the Yang-Mills matter is repulsive.

However, in process to obtain the above result, all surface terms are droped. In the other words, $T^\mu_\mu = 0$ is only a result in the case of topological trivial. In static case, if a non-trivial topological effect exists here:

$$\int d^3\vec{x}\partial_i(\bar{F}^a_{i0}A^a_\nu) = \text{constant}, \tag{1}$$

the conformal invariance of pure Yang-Mills theory is obviously broken, where $\bar{F}^a_{\mu\nu}(\vec{x}) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu 2g\epsilon^{abc}A^b_\mu A^c_\nu$, $(a = 1, ..., N^2-1)$ are SU(N) Yang-Mills field strength. In this present paper, the "static" means that all gauge fields and gauge transformations are time-indepenedent. The broken of conformal invariance can be confirmed simply by a scale transformation $\vec{x} \to \vec{x}/\lambda$: The right side of eq. (1) implies that the left side of eq. (1) should be invariant at $|x| \to \infty$. It conflicts with the conformal invariance of pure Yang-Mills theory, which requires that left side of eq. (1) should be proportional to λ or vanish at $|x| \to \infty$.

Let us further clarify what role this non-trivial topological effect plays. Noether's theorem yields conservation stress-energy tensor as follow

$$T_{\mu\nu} = \frac{1}{4}\eta_{\mu\nu}F^a_{\alpha\beta}F^{\alpha\beta a} - F^a_{\mu\beta}\partial_\nu A^{\beta a}. \tag{2}$$

When topology is trivial and Yang-Mills field is no-singular, a surface term can be freely added in stress-energy tensor for obtaining gauge invariant form,

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$$T_{\mu\nu} \to T_{\mu\nu} + \partial^\lambda (F^a_{\mu\lambda} A^a_\nu) = \frac{1}{4}\eta_{\mu\nu} F^a_{\alpha\beta} F^{\alpha\beta a} - F^a_{\mu\beta} F^{\mu\beta a}. \tag{3}$$

Eq. (3) leads to $T^\mu_\mu = 0$ in four-dimensional spacetime. However, for the case of non-trivial topology presence(eq. (1)), this surface term can not be added, since it obviously changes physical energy-momentum. So that $T^\mu_\mu = 0$ can not be yielded in this case.

When Yang-Mills fields are static, the Coleman-Deser equation [1,2,5]

$$2\int d^3\vec{x} T^i_i = \int d^3\vec{x}\{\frac{1}{2}\bar{F}^a_{ij}\bar{F}^a_{ij} + \bar{F}^a_{i0}\bar{F}^a_{i0}\} = 0 \tag{4}$$

can be obtained even though the non-trivial topology is present, since we need only static Yang-Mills equations and requirement of finite energy to obtain it . This equation provides some constrains for static solution of Yang-Mills equations. For trivial topology, the term $\bar{F}^a_{i0}\bar{F}^a_{i0}$ vanishes due to static Yang-Mills equations. It leads to the trivial solution $\bar{F}^a_{i0} = \bar{F}^a_{ij} = 0$. However, for non-trivial topology, $\bar{F}^a_{i0}\bar{F}^a_{i0}$ does not vanish. Then Coleman-Deser equation admits static solution of pure Yang-Mills theory when $\bar{F}^a_{ij}\bar{F}^a_{ij} < 0$ or $\bar{F}^a_{i0}\bar{F}^a_{i0} < 0$. Meanwhile, the static Hamilton density

$$\mathcal{H} = \frac{1}{4}\bar{F}^a_{ij}\bar{F}^a_{ij} - \frac{1}{2}\bar{F}^a_{i0}\bar{F}^a_{i0} \tag{5}$$

implies that only $\bar{F}^a_{i0}\bar{F}^a_{i0} < 0$ is allowed. It indicates that the static solution exists only for imaginary A^a_0. It means that A^a_0 have to be continued to complex plane analytically. The complex solutions were studied by Dolan [6] in Euclidian space. He obtained some Abelian-like solutions with zero action. In this present paper, we will actually study imaginary solution of A_0 with in Minkowski space. It will yield finite, no-zero energy. In the static pure Yang-Mills theory, at least, this imaginary A^a_0 is allowed by all fundamental principle, such as positvity of energy, hermitian of Hamilton, etc..

In the following we will try to find an analytic, static, non-singular soliton soultion of source-free SU(2) Yang-Mills equations. We can define "static dual" in Minkowski space,

$$\tilde{\bar{F}}^a_{ij} = i\epsilon_{ijk}\bar{F}^a_{k0}, \qquad \tilde{\bar{F}}^a_{k0} = -\frac{i}{2}\epsilon_{ijk}\bar{F}^a_{ij}. \tag{6}$$

where $a = 1, 2, 3$. The "static dual" in the above equation satisfy

$$\tilde{\tilde{\bar{F}}}^a_{ij} = \bar{F}^a_{ij}, \qquad \tilde{\tilde{\bar{F}}}^a_{i0} = \bar{F}^a_{i0}. \tag{7}$$

It is easily to check that, the static Yang-Mills equations and Coleman-Deser equation (4) will be automatically statisfied if field strengths are "static self-dual"

$$\bar{F}^a_{\mu\nu} = \tilde{\bar{F}}^a_{\mu\nu}. \tag{8}$$

In addition, the "static self-dual" also yielded minimum energy of the system. [1]

For obtaining an explicit analytic solution, in this paper we take a spherical sysmmetry ansatz for gauge fields

$$\begin{cases} A_{ia} = \dfrac{1}{g}\dfrac{f(r)}{r}\epsilon_{ian}\hat{x}_n, \\ A_{0a} = \dfrac{i}{g}\phi(r)\hat{x}^a, \end{cases} \qquad \hat{x} = \frac{x_a}{r}. \tag{9}$$

Than eqs. (8) and (9) lead to

[1]The solutions of eq. (8) may be nameed as "static instanton". However, it must be pointed out that the usual static instanton is not well-defined: The instanton is related to tunneling effects of quantum mechanics which exsits only in Euclidean space (imaginary time). In the case absence time, there exists no tunneling effects, so that solutions of eq (8) is different from traditional instanton.

$$
\begin{cases} f' = -\phi(1-2f), \\ \phi' = -\dfrac{2}{r^2} f(1-f), \end{cases} \tag{10}
$$

with $f' = \frac{d}{dr}f$, $\phi' = \frac{d}{dr}\phi$. To obtain the above equation, the identity,

$$
\epsilon_{jan}\hat{x}_i\hat{x}_n - \epsilon_{ian}\hat{x}_j\hat{x}_n + \epsilon_{ijn}\hat{x}_a\hat{x}_n = \epsilon_{ija}, \tag{11}
$$

has been used. The eq. (10) can reduce to static Liouville equation [7]. Its solutions are well-know. An analytic, non-sigular solution is,

$$
\begin{cases} f = \dfrac{1}{2}(1 - \dfrac{\kappa r}{\sinh(\kappa r)}), \\ \phi = \dfrac{1}{2r}(1 - \dfrac{\kappa r}{\tanh(\kappa r)}), \end{cases} \tag{12}
$$

where κ is a positive integral constant with mass-dimension. The solutions (12) lead to their asymptotic behaviour as follows

$$
\begin{array}{ll} f \xrightarrow{r\to 0} \dfrac{1}{12}\kappa^2 r^2, & \phi \xrightarrow{r\to 0} -\dfrac{1}{6}\kappa^2 r, \\ f \xrightarrow{r\to\infty} \dfrac{1}{2} - e^{-\kappa r}, & \phi \xrightarrow{r\to\infty} -\dfrac{\kappa}{2} + \dfrac{1}{2r}. \end{array} \tag{13}
$$

Thus it is suprised that $A_0 \to$ constant instead of zero at $r \to \infty$ (it is different from instanton and monopole), but while all $F_{\mu\nu}$ still fall off as r^{-2}..

Due to "static self-dual" of field strength, the topological mass (energy) of the soliton is

$$
\begin{aligned} M &= \int d^3\vec{x}\mathcal{H} = -\int d^3\vec{x}\bar{F}^a_{i0}\bar{F}^a_{i0} = -\int d^3\vec{x}\partial_i(A^a_0\bar{F}^a_{i0}) \\ &= -\int d^3\vec{x}\partial_i(A^a_0\partial_i A^a_0) = \frac{\pi}{g^2}\kappa. \end{aligned} \tag{14}
$$

Therefore, the integral constant κ can be interpreted as an unit of energy. Moreover, we can see that the topological effect in our solution is characterized by the topological mass, which is determined by asympotic behaviour of A^a_0 at $r \to \infty$. The energy density distribution function is

$$
\rho(r) = -\bar{F}^a_{i0}\bar{F}^a_{i0} = g^{-2}\{\phi\phi'' + \frac{2}{r}\phi\phi' + \phi'^2\}. \tag{15}
$$

The energy density distribution is shown in fig. 1. Effective radius of the soliton (soliton size) is usually defined as half maximum width of spectral distribution. Then From the fig. 1 we can see that the effective radius of this soliton is $r_0 = 1.3\kappa^{-1}$. It is interesting that $r_0 M$ is κ-independent. The fig. 1 also shows that this soliton is stable for any boundary conditions, i.e., fixed κ.

We have shown that there exist static topological solution with finite energy in pure Yang-Mills theory. This fact indicates that the massless gauge particles in Yang-Mills theory can become static, massive particles due to non-linear self-interaction. It is interesting to compare with U(1) electromagnetic theory, in which the photon can not be static and massive forever. In our solution, the zero component of Yang-Mills field is imaginary. However, it does not cause any problem for static pure Yang-Mills theory. In fact, the ansatz (9) naturally satisfies the Coulomb gauge condition, $\partial_i A_i = 0$. Then for the static case, Lorentz condition $\partial_\mu A^\mu = 0$ allows A_0 to be arbitrary time-independent function (no matter what it is real or imaginary).

The confinement of gauge particles is an active subject (specially in quantum chromodynamics). Although the solution (12) yields finite energy, the confinement of gauge particles is still allowed here. There are two different mechanisms to achieve the confinement: The first one is to take a limit of solution (12), i.e., $\kappa \to \infty$. It is obvious that, in this limit topological mass $M \to \infty$ and effective radius $r_0 \to 0$. It means that the static point-like particles will generate infinite energy. The second one is that there exist some other singular solutions for eq. (12). For example, the solution

$$
f = r\phi = \frac{\lambda}{r + 2\lambda} \tag{16}
$$

leads gauge fields to be singular at $r = 0$. Thus energy obtained from solution (16) is infinite. These two mechanisms can be distinguished by asymptotic behaviour of ϕ (or A_0) at $r \to \infty$. In the first one, $\phi \to \infty$ and $\phi' \to 0$ at $r \to \infty$ when $\kappa \to \infty$. Meanwhile, in the second one, $\phi \to r^{-2} \to 0$ at $r \to \infty$. The above discussion indicates that both of confinement and no-confinement are allowed in pure Yang-Mills theory. Whether or not presence of confinement is determined by asymptotic behaviour of A_0^a at $r \to \infty$.

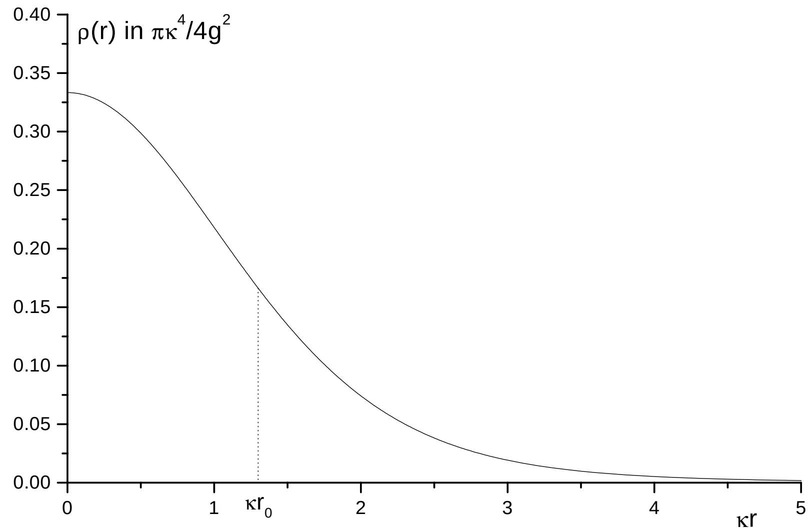

FIG. 1. The density distribution of topological soliton. It implies that the effective radius(width of wave packet) is $r_0 = 1.3\kappa^{-1}$.

To conclude, we obtain a static spherical sysmmetry no-singular soliton solution in source-free Yang-Mills theory when a non-trivial topology is present. This topological soliton is stable, and has finite, positive and minimum mass (energy). Value of the topological mass can be fixed by effective radius of the solution. At the limit of point-particle, the gauge fields will be confinement (with infinite energy). The topological effect is determined by asymptotic behaviour of A_0^a at $r \to \infty$.

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